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Reggeization in high energy QCD*

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Abstract: We study QCD in the regime of high parton density arising in hadronic collisions at large center-of-mass energy. The n-gluon amplitudes of the generalized leading logarithmic approximation are investigated. We find identities relating amplitudes with different numbers of gluons in the t-channel. These identities constrain the reggeization of the gluon in high energy QCD. The tensors describing the reggeization of a gluon in color space are identified.

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1. Introduction

A longstanding problem in the physics of strong interactions is to understand the Regge limit of hadronic scattering amplitudes. Especially challenging is the description of this kinematical region of large center-of-mass energy $\sqrt{s}$ and small momentum transfer $\sqrt{t}$ in terms of quark and gluon degrees of freedom. Regge theory provides a successful phenomenology but it has not yet been derived from QCD. The main problem is the fact that hadronic scattering in the Regge limit is in general characterized by large parton densities and small momentum scales. A full understanding will eventually require non-perturbative methods. Scattering processes involving small color dipoles, however, can be described using perturbative methods. Examples are heavy onium scattering or collisions of highly virtual photons. These processes involve a large momentum scale (the heavy quark mass or the photon virtuality, respectively) and can be treated perturbatively even at high energy. We therefore hope that by studying these processes some essential features of the dynamics of high energy QCD can be discovered using perturbative methods.

The starting point of the perturbative approach to high energy QCD is the BFKL Pomeron [1, 2]. It resums terms of the form $(\alpha_s \log s)^n$ in which large logarithms of the energy can compensate the smallness of the strong coupling constant α_s. The corresponding approximation scheme is known as the leading logarithmic approximation (LLA). The BFKL Pomeron leads to a power-like growth of total cross sections with energy, $\sigma \sim s^{\lambda}$, where $\lambda = 4\alpha_s N_c \ln 2/\pi$. On the level of the total cross section this perturbative Pomeron can be interpreted as an exchange of a ladder of gluons in the t-channel. In other words: the BFKL equation describes the exchange of a bound state of two gluons in the t-channel.

A very important outcome of the perturbative (BFKL) analysis of high energy QCD is the so-called reggeization of the gluon [3]. Technically speaking it manifests itself in the fact that the BFKL equation in the color octet channel exhibits a pole solution, corresponding to a single (reggeized) particle with gluon quantum numbers propagating in the t-channel. Looking at this fact from another angle, we see that a gluon propagating in the t-channel turns out to be a composite object, namely a bound state of two gluons. This apparently confusing situation becomes quite natural when we realize that elementary gluons are just not the most suitable degrees of freedom. Instead, the natural degrees of freedom in the high energy limit are collective excitations of the non-abelian gauge field, i.e. the reggeized gluons or reggeons. The elementary gluon and the bound state of two gluons in a color octet state mentioned above can then be interpreted as the first two terms in an expansion of the reggeized gluon in the number of its constituents, similar to the Fock states of a usual bound state. It is the purpose of the present paper to study this expansion in more detail. We will in particular investigate the behavior of color quantum numbers in the process of reggeization.

The most suitable framework for studying these aspects of reggeization is the generalized leading logarithmic approximation (GLLA) [4]–[6] which is a natural extension of the BFKL formalism to larger numbers of gluons in the t-channel. It has been widely studied in the context of unitarity corrections. These unitarity corrections and the GLLA have mostly been analysed for the process of virtual photon collisions, but the results are expected to apply to a variety of processes. In collisions at very large energies the parton densities become large. Consequently, parton recombination effects become important and the rapid growth of cross sections with energy will be slowed down. The GLLA has been devised to incorporate these recombination effects. It is most conveniently formulated in terms of multi-gluon amplitudes describing the production of a number of gluons in the t-channel. During the evolution in the t-channel the number of gluons is allowed to fluctuate and thus transitions between different n-gluon states are possible. The two-gluon amplitude in this framework is just the usual BFKL amplitude.

The amplitudes for the production of up to four gluons in the t-channel have been analysed in [7]–[12]. They turn out to have a very interesting field theory structure which manifests itself in the following way. In these amplitudes there is a two-gluon state (the BFKL Pomeron) as well as a state of four interacting gluons in the t-channel. In addition, there is a transition vertex $V_{2\to 4}$ coupling these

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states to each other. The latter is a number-changing vertex and thus turns the quantum mechanical system of n-gluon states into a quantum field theory of interacting n-gluon states. Interestingly, only states of even numbers of gluons occur. The amplitudes are functions of the two-dimensional transverse momenta of the gluons. They also depend on rapidity which can be understood as a time-like variable. After a Fourier transformation to two-dimensional impact parameter space the amplitudes can be shown to be invariant under conformal transformations of the gluon coordinates [13, 14]. In [15] also the amplitudes with up to six gluons have been studied. It was shown that the field theory structure persists also to larger numbers of gluons and the transition vertex from a two-gluon to a six-gluon state was calculated. Also this vertex is conformally invariant in impact parameter space [16]. In summary, the n-gluon amplitudes have the structure of a conformally invariant field theory in two-dimensional impact parameter space with rapidity as an additional time-like parameter. In its present form this field theory is formulated in terms of n-gluon states and vertices. It would obviously be desirable to extract from these the necessary information to identify the corresponding two-dimensional conformal field theory and to apply the powerful methods available for the latter. At present this important step is an open problem. A good understanding of how the field theory structure emerges should be helpful to make progress in this direction. We expect reggeization to play a significant rôle in this context because it has been found to be an indispensable condition for the emergence of the field theory structure [15]. Especially the details of reggeization in color space are expected to contain important information.

The n-gluon amplitudes have been investigated for up to six t-channel gluons. Moreover, a certain part of the amplitudes can be computed even for arbitrary n, the result being a superposition of BFKL amplitudes. These results are explained in some detail in section 2 where we also collect some formulae and notation needed in this paper. Equipped with these (partial) solutions we are in a position to study two novel aspects of their structure related to reggeization. Both concern the intricate interplay of color and momentum structure present in the amplitudes. The first aspect is the subject of section 3. It is of a more global nature and relates n-gluon amplitudes of different n. The corresponding identities are obtained when one of the gluon momenta vanishes. They place strong constraints on the color structure of the complete amplitude. In section 4 we encounter more local properties, namely the color tensors accompanying the reggeization of a single gluon. Our aim is to extract from the known amplitudes as much information as possible. We then formulate conjectures on how the observations made here can be generalized to a larger number of t-channel gluons or to the unknown pieces of the amplitudes, respectively. We also discuss the relevance and the potential uses of the conjectures for the further investigation of the effective field theory of unitarity corrections.

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2. Multi-gluon amplitudes and reggeization

In this section we collect results which we will need in the subsequent sections. For a more extensive review and the detailed derivation of these results the reader is referred to [15] and references therein.

2.1 BFKL equation and reggeization

The BFKL equation for the partial wave amplitude ϕ_ω reads

$$\omega\phi_\omega(\mathbf{k}_1,\mathbf{k}_2) = \phi_0(\mathbf{k}_1,\mathbf{k}_2) + \int \frac{d^2\mathbf{l}}{(2\pi)^3}\,\frac{1}{\mathbf{l}^2(\mathbf{q}-\mathbf{l})^2} K_{\mathrm{BFKL}}(\mathbf{l},\mathbf{q}-\mathbf{l};\mathbf{k}_1,\mathbf{k}_2)\,\phi_\omega(\mathbf{l},\mathbf{q}-\mathbf{l})\,. \tag{2.1}$$

Here $\mathbf{k}_1$ and $\mathbf{k}_2$ are the transverse momenta of the two gluons in the t-channel, and $q = \mathbf{k}_1 + \mathbf{k}_2$ is the total transverse momentum transfer in the t-channel. ω denotes the complex angular momentum conjugate to rapidity. ϕ_0 is an inhomogeneous term depending on the process under consideration which couples the two gluons to external particles. K_{BFKL} is the Lipatov kernel,

$$\begin{aligned} K_{\mathrm{BFKL}}(\mathbf{l},\mathbf{q}-\mathbf{l};\mathbf{k},\mathbf{q}-\mathbf{k}) = -N_c g^2 \left[\mathbf{q}^2 - \frac{\mathbf{k}^2(\mathbf{q}-\mathbf{l})^2}{(\mathbf{k}-\mathbf{l})^2} - \frac{(\mathbf{q}-\mathbf{k})^2\mathbf{l}^2}{(\mathbf{k}-\mathbf{l})^2}\right] + \\ +(2\pi)^3\mathbf{k}^2(\mathbf{q}-\mathbf{k})^2\,[\,\beta(\mathbf{k})+\beta(\mathbf{q}-\mathbf{k})]\,\delta^{(2)}(\mathbf{k}-\mathbf{l}) \end{aligned} \tag{2.2}$$

with the gluon trajectory function $\alpha(\mathbf{k}^2) = 1 + \beta(\mathbf{k}^2)$ where

$$\beta(\mathbf{k}^2) = -\frac{N_c}{2}g^2 \int \frac{d^2\mathbf{l}}{(2\pi)^3}\frac{\mathbf{k}^2}{\mathbf{l}^2(\mathbf{l}-\mathbf{k})^2}\,. \tag{2.3}$$

If the two gluons in the t-channel are in a color octet state the factor $-N_c$ in the kernel (2.2) has to be replaced by $-N_c/2$. Assuming that ϕ_0 depends only on the sum $q = \mathbf{k}_1 + \mathbf{k}_2$ of the momenta of the two t-channel gluons we find that the BFKL equation (2.1) has a pole solution

$$\phi^{\mathbf{8_A}}(\mathbf{k}_1+\mathbf{k}_2) = \frac{\phi_0^{\mathbf{8_A}}(\mathbf{k}_1+\mathbf{k}_2)}{\omega - \beta(\mathbf{k}_1+\mathbf{k}_2)}\,. \tag{2.4}$$

As already discussed in the Introduction it signals the reggeization of the gluon, i.e. it indicates that the color octet exchange is a composite object of two gluons. In addition, the amplitude $\phi^{\mathbf{8_A}}$ can be ‘squared’ in the t-channel to give a result proportional to the trajectory function $\beta(\mathbf{q}^2)$ (see (2.3)) of the reggeized gluon. In the present paper, however, we will not make use of the corresponding t-channel unitarity relations and concentrate on the amplitudes describing the production of 2 (or in general n) gluons in the t-channel.

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2.2 Multigluon amplitudes

The multigluon amplitudes D_n we will consider from now on apply to the process of virtual photon-photon scattering. They are cut in the t-channel and thus describe the production of n gluons in the t-channel starting from two virtual photons. Similar to the BFKL equation they are partial wave amplitudes depending on the complex angular momentum ω (which we suppress in the notation). They are also cut in the $n-1$ energy variables formed from the first photon and the i first gluons ($1 \leq i < n$). The amplitudes depend on the n transverse momenta $\mathbf{k}_i$ of the gluons and on their color labels a_i. These amplitudes D_n are obtained as solutions of a tower of coupled integral equations. The equation for $n = 2$ is just the BFKL equation with the particular inhomogeneous term induced by the coupling of two gluons to the virtual photons. In [15] the integral equations have partly been solved. Before giving the solutions explicitly we first introduce suitable tensors and notation in color space.

We consider the gauge group $\mathrm{SU}(N_c)$ with generators t^a ($a = 1, \ldots, N_c^2 - 1$). The algebra is

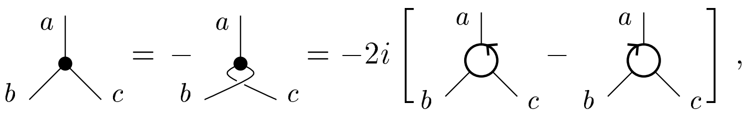
$$[t^a, t^b] = i f_{abc} t^c \,. \tag{2.5}$$

and for the structure constants f_{abc}

$$f_{abc} = -f_{acb} = -2i \left[\mathrm{tr}(t^a t^b t^c) - \mathrm{tr}(t^c t^b t^a) \right] . \tag{2.6}$$

Introducing birdtrack notation [17] they become

$$f_{abc} =$$



(2.7)

in which the orientated lines denote quark color representations whereas the unorientated lines correspond to gluon color lines. It will be useful to define the color tensors

$$d^{b_1 b_2 \ldots b_n} = \mathrm{tr}(t^{b_1} t^{b_2} \ldots t^{b_n}) + \mathrm{tr}(t^{b_n} \ldots t^{b_2} t^{b_1}) \tag{2.8}$$

$$f^{b_1 b_2 \ldots b_n} = \frac{1}{i} \left[\mathrm{tr}(t^{b_1} t^{b_2} \ldots t^{b_n}) - \mathrm{tr}(t^{b_n} \ldots t^{b_2} t^{b_1}) \right] . \tag{2.9}$$

These tensors are obviously invariant under cyclic permutations of the color labels. To make contact with the usual structure constants (which we write with lower indices) we mention that $d^{abc} = \frac{1}{2} d_{abc}$ and $f^{abc} = \frac{1}{2} f_{abc}$. Further we have $d^{ab} = \delta_{ab}$.

It will also be useful to recall some facts about invariant tensors. The n gluons of our amplitudes D_n form an overall color singlet. They can therefore be expanded in color space into a linear combination of invariant $\mathrm{su}(N_c)$ tensors. Invariant tensors in a simple Lie algebra can generally be constructed from traces of group generators,

$$\Theta^{a_1 \ldots a_n} = \mathrm{tr}(t^{a_1} \ldots t^{a_n}) \,. \tag{2.10}$$

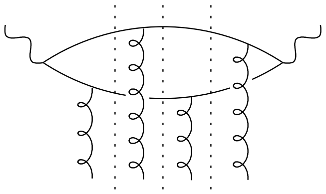


Figure 1: Cut amplitude contributing to the coupling of $n = 4$ gluons to a quark loop.

For such a trace the property of being an invariant tensor reads

$$\sum_{i=1}^{n} f_{ca_i b}\, \Theta^{a_1 \dots b \dots a_n} = 0\,, \tag{2.11}$$

t^b being inserted at the ith position in the tensor (i.e. in the trace). Equation (2.11) is often used as the defining property of an invariant tensor. Our tensors defined in eqs. (2.8) and (2.9) are obtained as sum or difference of such traces and thus invariant tensors fulfilling the condition (2.11).

The integral equations describing the n-gluon amplitudes contain as a lowest order term the coupling $D_{(n;0)}$ of n gluons to the two virtual photons via a quark loop, as shown in figure 1 for the case of four gluons.

The n gluons are attached to the quark loop in all possible ways in order to preserve gauge invariance, but due to the cuts (indicated in the figure by dashed lines) the gluons are not allowed to cross in the t-channel. The term $D_{(n;0)}$ is thus obtained as the sum of 2^n diagrams since each of the gluons can be coupled either to the quark or to the antiquark. As discussed in detail in [15] the lowest order amplitude $D_{(n;0)}$ can be expressed in terms of $D_{(2;0)}$ only. The coupling of three gluons to the quark loop is for example given by

$$D_{(3;0)}^{a_1 a_2 a_3}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = \frac{1}{2} g f_{a_1 a_2 a_3} \left[D_{(2;0)}(12,3) - D_{(2;0)}(13,2) + D_{(2;0)}(1,23) \right]. \tag{2.12}$$

Here we have introduced a new notation for the momentum arguments: for brevity we replace the momentum $\mathbf{k}_i$ by its index, and a string of indices stands for to the sum of the corresponding momenta such that for example

$$D_{(2;0)}(12,3) = D_{(2;0)}(\mathbf{k}_1 + \mathbf{k}_2, \mathbf{k}_3)\,. \tag{2.13}$$

The fact that in $D_{(3;0)}$ the eight diagrams reduce to three terms only is due to the fact that two of the diagrams play a special rôle: the diagrams in which all gluons are coupled to the same quark (or antiquark) line act as regularization terms only and do not lead to an extra term in (2.12).

We now turn to the full n-gluon amplitudes obtained as solutions of the integral equations. As already mentioned the two gluon amplitude D_2 is just the well-known

BFKL amplitude with the photon impact factor as the initial condition of the t-channel evolution. The color structure of D_2 is trivial,

$$D_2^{a_1 a_2}(\mathbf{k}_1, \mathbf{k}_2) = \delta_{a_1 a_2} D_2(\mathbf{k}_1, \mathbf{k}_2)\,. \tag{2.14}$$

In the following we will often make use of the latter function $D_2(\mathbf{k}_1, \mathbf{k}_2)$ (without color labels) denoting the momentum part of the full D_2. The amplitude for the production of three gluons in the t-channel is a superposition of two-gluon amplitudes,

$$D_3^{a_1 a_2 a_3}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = \frac{1}{2} g f_{a_1 a_2 a_3} \left[D_2(12, 3) - D_2(13, 2) + D_2(1, 23) \right]. \tag{2.15}$$

This is a generalization of the concept of reggeization described above for the two-gluon amplitude. There we found that the two-gluon amplitude in the color octet channel has the analytic properties of a one-gluon amplitude, see (2.4). In a similar way the three gluon amplitude reduces to two-gluon amplitudes. In each of the three terms in (2.15) two gluons arrange in such a way that they behave as a single gluon. Also here this happens in the color octet channel, but we will see later that reggeization can also take place in other color channels.

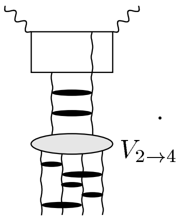
It can be shown even for arbitrary n that the n-gluon amplitude D_n contains a part that is a superposition of two-gluon amplitudes D_2. In the case of the three-gluon amplitude this so-called reggeizing part D_n^R exhausts the amplitude, whereas for higher n further parts have to be added, $D_n = D_n^R + D_n^I$. The solution for D_3 in (2.15) is very similar to the expression for the quark loop with three gluons attached, cf. eq. (2.12). It can be obtained from the latter by replacing the lowest order amplitude $D_{(2;0)}$ by the full two-gluon amplitude D_2 while keeping the color structure and the momentum arguments. The same procedure allows one to obtain the reggeizing part D_n^R from the lowest order coupling of n gluons to a quark loop $D_{(n;0)}$ even for arbitrary n. The quark loop in turn can be computed from the 2^n corresponding diagrams as described above. For later use we will now give the explicit expressions for the reggeizing parts D_n^R of the n-gluon amplitudes for n up to 5,

$$\begin{aligned}
D_4^{R\,a_1 a_2 a_3 a_4}&(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) = \\
&= -g^2 d^{a_1 a_2 a_3 a_4} \left[D_2(123, 4) + D_2(1, 234) - D_2(14, 23) \right] - \\
&\quad - g^2 d^{a_2 a_1 a_3 a_4} \left[D_2(134, 2) + D_2(124, 3) - D_2(12, 34) - D_2(13, 24) \right], \qquad (2.16)\\
D_5^{R\,a_1 a_2 a_3 a_4 a_5}&(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4, \mathbf{k}_5) = \qquad (2.17)\\
&= -g^3 \{ f^{a_1 a_2 a_3 a_4 a_5} \left[D_2(1234, 5) + D_2(1, 2345) - D_2(15, 234) \right] + \\
&\qquad + f^{a_2 a_1 a_3 a_4 a_5} \left[D_2(1345, 2) - D_2(12, 345) + D_2(125, 34) - D_2(134, 25) \right] + \\
&\qquad + f^{a_1 a_2 a_3 a_5 a_4} \left[D_2(1235, 4) - D_2(14, 235) + D_2(145, 23) - D_2(123, 45) \right] + \\
&\qquad + f^{a_1 a_2 a_4 a_5 a_3} \left[D_2(1245, 3) - D_2(13, 245) + D_2(135, 24) - D_2(124, 35) \right] \}\,.
\end{aligned}$$

The explicit expression for $n = 6$ can be found in [15].

The remaining parts D_n^I of the n-gluon amplitudes have been termed 'irreducible', because they cannot be reduced to two-gluon amplitudes. These parts are known explicitly for $n = 4$ and $n = 5$. In the case of the four-gluon amplitude D_4^I has the following structure[1] (here we suppress in the notation the dependence on the color labels and momenta),

$$D_4^I = G_4 \cdot V_{2\to 4} \cdot D_2 = \quad . \tag{2.18}$$



It starts with a two-gluon state coupled to the virtual photons via a quark loop (forming together D_2). Then there is a transition vertex $V_{2\to 4}$ coupling the two-gluon state to a four-gluon state in the t-channel. The vertex is known explicitly, but we will not need its explicit form here. The four-gluon state G_4 is not known analytically so far, but some important properties can be derived from its defining integral equation. One of these properties is that D_4^I vanishes if one of the four gluon momenta vanishes,

$$D_4^{I\,a_1a_2a_3a_4}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)\big|_{\mathbf{k}_i=0} = 0 \qquad (i \in \{1,\ldots,4\})\,. \tag{2.19}$$

The 'irreducible' part D_5^I of the five-gluon amplitude has been found to be a superposition of irreducible four-gluon amplitudes D_4^I,

$$\begin{aligned}
D_5^{I\,a_1a_2a_3a_4a_5}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4,\mathbf{k}_5) &= \frac{g}{2}\times \\
\times \big\{ f_{a_1a_2c} D_4^{I\,ca_3a_4a_5}&(12,3,4,5) + f_{a_1a_3c} D_4^{I\,ca_2a_4a_5}(13,2,4,5) + \\
+ f_{a_1a_4c} D_4^{I\,ca_2a_3a_5}&(14,2,3,5) + f_{a_1a_5c} D_4^{I\,ca_2a_3a_4}(15,2,3,4) + \\
+ f_{a_2a_3c} D_4^{I\,a_1ca_4a_5}&(1,23,4,5) + f_{a_2a_4c} D_4^{I\,a_1ca_3a_5}(1,24,3,5) + \\
+ f_{a_2a_5c} D_4^{I\,a_1ca_3a_4}&(1,25,3,4) + f_{a_3a_4c} D_4^{I\,a_1a_2ca_5}(1,2,34,5) + \\
+ f_{a_3a_5c} D_4^{I\,a_1a_2ca_4}&(1,2,35,4) + f_{a_4a_5c} D_4^{I\,a_1a_2a_3c}(1,2,3,45) \big\}\,.
\end{aligned} \tag{2.20}$$

The fact that D_5^I reduces to a sum of four-gluon amplitudes already indicates that the name 'irreducible' needs more explanation. In fact there is some deeper insight hidden here, and we will discuss this issue in detail in section 3.3. In [15] an integral equation for the corresponding part D_6^I of the six-gluon amplitude has been derived. Unfortunately, it has not yet been possible to solve it completely. But there are strong indications that the six-gluon amplitude consists of a reggeizing part, a part reducing to four-gluon amplitudes D_4^I and a third part that contains a full six-gluon state, and the corresponding 2-to-6 gluon vertex has already been identified.

[1] Here G_4 and $V_{2\to 4}$ should be understood as integral operators, for the details see again [15].

3. Ward type identities

The Ward type identities to be discussed in this section relate n-gluon amplitudes of different n and allow us to gain further insight into the interplay between their color and momentum structure. These identities of Ward type arise when we set one of the n transverse momenta $\mathbf{k}_i$ to zero. Roughly speaking, the amplitude D_n can in this case be expressed in terms of the amplitude D_{n-1}. The reduction is accompanied by an interesting behaviour of the corresponding color structure for which we can extract a general rule. This behavior in color space does not only involve the gluons to which the one with vanishing momentum is coupled in the original amplitude (for example in the reggeizing parts) but involves all gluons of the amplitude. In this sense our identities constitute a global property of the amplitudes.

We will be able to find a formula valid for the reggeizing parts D_n^R of the amplitudes and for these we can even give a general proof for arbitrary n. For the remaining parts D_n^I we limit our study to the cases $n = 4, 5$ for which these parts are known explicitly. We will then make a conjecture on how the the mechanism works for higher n here. If it can be confirmed, the identities might provide a valuable tool for the further investigation of the field theory structure of unitarity corrections. Specifically, we will find a characteristic difference between the parts of the amplitudes that exhibit reggeization and such parts that do not. Moreover, the identities lead to strong constraints on the amplitudes. Both facts might turn out to be very helpful especially for the investigation of higher n-gluon amplitudes with $n \geq 6$ where a more complicated structure is expected to arise. We will discuss the potential significance of the Ward type identities for the field theory structure in more detail in subsection 3.3.

We start with considering the reggeizing parts D_n^R and study for each n how the color tensors rearrange in the case of a vanishing momentum $\mathbf{k}_i$. It seems to us quite instructive to see the mechanism at work in concrete examples. With these we also hope to convey the impression that the Ward type identities impose very strong constraints on the color and momentum structure of the amplitudes. After that we state the general rule for the amplitudes D_n^R in (3.18), (3.19) and sketch the proof for arbitrary n. Then we turn to the amplitudes D_4^I and D_5^I and formulate the conjecture how higher D_n^I have to be treated.

3.1 The reggeizing parts D_n^R

After recalling that the BFKL amplitude D_2 vanishes if one of its momentum arguments vanishes,

$$D_2(\mathbf{k}_1, \mathbf{k}_2)|_{\mathbf{k}_1=0} = D_2(\mathbf{k}_1, \mathbf{k}_2)|_{\mathbf{k}_2=0} = 0\,, \tag{3.1}$$

we consider first the reggeizing parts D_n^R of the amplitudes D_n, with n ranging from 3 to 5. This includes also the full amplitude D_3 since it consists of reggeizing pieces only. The simplest relations hold for the case in which we set the first momentum $\mathbf{k}_1 = 0$, namely the vanishing of the amplitudes,[2]

$$D_3\big|_{\mathbf{k}_1=0} = D_4^R\big|_{\mathbf{k}_1=0} = D_5^R\big|_{\mathbf{k}_1=0} = 0\,. \tag{3.2}$$

The same is true for setting the nth (i.e. the last) momentum to zero in the amplitude D_n^R,

$$D_3\big|_{\mathbf{k}_3=0} = D_4^R\big|_{\mathbf{k}_4=0} = D_5^R\big|_{\mathbf{k}_5=0} = 0\,. \tag{3.3}$$

We will see below that the identities (3.2) and (3.3), although seemingly trivial, fit well into the more general rule that determines the color structure of our Ward type identities. When setting one of the momenta $\mathbf{k}_2, \ldots, \mathbf{k}_{n-1}$ to zero the amplitudes do not vanish. For the three-gluon amplitude we find

$$\begin{aligned} D_3^{a_1a_2a_3}\big|_{\mathbf{k}_2=0} &= g f_{a_1a_2a_3} D_2(\mathbf{k}_1, \mathbf{k}_3) \\ &= g \left[\,\underset{\curlywedge}{|}\;|\;\right] \star D_2^{b_1b_2}(\mathbf{k}_1, \mathbf{k}_3) = g \left[\;|\;\underset{\curlywedge}{|}\,\right] \star D_2^{b_1b_2}(\mathbf{k}_1, \mathbf{k}_3)\,. \end{aligned} \tag{3.4}$$

In the second line we have used the 2-gluon amplitude including color labels, that is $D_2^{b_1b_2} = \delta_{b_1b_2} D_2$ (see (2.14)). The way the color structure is written in the second line serves to make the general rule for the color structure more transparent. Here we have slightly extended the birdtrack notation by defining

$$\left[\,\underset{\curlywedge}{|}\;|\;\right] \star \Theta^{\{b\}} = \overset{\{b\}}{\underset{\{a\}}{\underset{\curlywedge}{|}\;|}} \star \Theta^{\{b\}} = f_{a_1a_2b_1}\delta_{a_3b_2}\Theta^{b_1b_2}\,, \tag{3.5}$$

where the symbol $\star$ stands for the contraction of the set $\{b\}$ of color labels. For the reggeizing part D_4^R of the four-gluon amplitude we find

$$\begin{aligned} D_4^{R\,a_1a_2a_3a_4}\big|_{\mathbf{k}_2=0} &= g f_{a_1a_2c} D_3^{ca_3a_4}(\mathbf{k}_1, \mathbf{k}_3, \mathbf{k}_4) &(3.6)\\ &= g \left[\,\underset{\curlywedge}{|}\;|\;|\;\right] \star D_3^{b_1b_2b_3}(\mathbf{k}_1, \mathbf{k}_3, \mathbf{k}_4)\,. &(3.7) \end{aligned}$$

This can be seen directly from D_4^R. For $\mathbf{k}_2 = 0$ the two expressions in square brackets in (2.16) become equal due to (3.1) and the color tensor in (3.6) is the difference of the two color tensors in (2.16),

$$d^{abcd} - d^{bacd} = -\frac{1}{2} f_{abk} f_{kcd}\,. \tag{3.8}$$

The color tensor corresponding to D_3, i.e. $f_{b_1b_2b_3}$, is an invariant tensor. According to (2.11) the second line in (3.6) can thus also be written as

$$D_4^{R\,a_1a_2a_3a_4}\big|_{\mathbf{k}_2=0} = g \left[\;|\;\underset{\curlywedge}{|}\;|\; + \;|\;|\;\underset{\curlywedge}{|}\,\right] \star D_3^{b_1b_2b_3}(\mathbf{k}_1, \mathbf{k}_3, \mathbf{k}_4)\,. \tag{3.9}$$

[2]Strictly speaking, we here make incorrect use of notation since the amplitudes D_n are for different n objects in different $\bigotimes_n[su(N_c)]$ tensor spaces.

For $\mathbf{k}_3 = 0$ we find in the same way

$$\begin{aligned} D_4^{R\,a_1a_2a_3a_4}\big|_{\mathbf{k}_3=0} &= g f_{a_3a_4c} D_3^{a_1a_2c}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_4) \\ &= g \Big[\quad \Big] \star D_3^{b_1b_2b_3}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_4) \\ &= g \Big[\quad + \quad \Big] \star D_3^{b_1b_2b_3}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_4)\,. \end{aligned} \tag{3.10}$$

The Ward identities for the reggeizing part D_5^R of the five-gluon amplitude arise in a similar way. Setting one of the outgoing momenta to zero in (2.17), we find that the four different momentum structures reduce to two. When $\mathbf{k}_2 = 0$, for instance, the expressions in square brackets in line 1 and 2 in (2.17) become equal up to a sign, as do the expressions in square brackets in line 3 and 4. The corresponding pairs of color tensors can be added using (2.5),

$$f^{a_1a_2a_3a_4a_5} - f^{a_2a_1a_3a_4a_5} = f_{a_1a_2c}\, d^{ca_3a_4a_5} \tag{3.11}$$

$$f^{a_1a_2a_3a_5a_4} - f^{a_1a_2a_4a_5a_3} = f_{a_1a_2c}\, d^{a_3ca_4a_5}\,. \tag{3.12}$$

Comparing with (2.16) we can thus write

$$\begin{aligned} D_5^{R\,a_1a_2a_3a_4a_5}\big|_{\mathbf{k}_2=0} &= g f_{a_1a_2c} D_4^{R\,ca_3a_4a_5}(\mathbf{k}_1, \mathbf{k}_3, \mathbf{k}_4, \mathbf{k}_5) \\ &= g \Big[\quad \Big] \star D_4^{R\,b_1b_2b_3b_4}(\mathbf{k}_1, \mathbf{k}_3, \mathbf{k}_4, \mathbf{k}_5) \\ &= g \Big[\quad + \quad + \quad \Big] \star D_4^{R\,b_1b_2b_3b_4}(\mathbf{k}_1, \mathbf{k}_3, \mathbf{k}_4, \mathbf{k}_5)\,. \end{aligned} \tag{3.13}$$

For $\mathbf{k}_4 = 0$ similarly

$$\begin{aligned} D_5^{R\,a_1a_2a_3a_4a_5}\big|_{\mathbf{k}_4=0} &= g f_{a_4a_5c} D_4^{R\,a_1a_2a_3c}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_5) \\ &= g \Big[\quad \Big] \star D_4^{R\,b_1b_2b_3b_4}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_5) \\ &= g \Big[\quad + \quad + \quad \Big] \star D_4^{R\,b_1b_2b_3b_4}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_5)\,. \end{aligned} \tag{3.14}$$

The last lines in (3.13) and (3.14) are again implied by the fact that the color tensors in $D_4^{R\,b_1b_2b_3b_4}$ are invariant tensors. For $\mathbf{k}_3 = 0$ we need the analogue of (3.11), (3.12) which is slightly more complicated. Applying (2.5) two times we get the two identities

$$f^{a_1a_2a_3a_4a_5} - f^{a_1a_2a_4a_5a_3} = f_{a_1a_3c}\, d^{ca_2a_4a_5} + f_{a_2a_3c}\, d^{a_1ca_4a_5} \tag{3.15}$$

$$f^{a_2a_1a_3a_4a_5} + f^{a_1a_2a_3a_5a_4} = f_{a_1a_3c}\, d^{a_2ca_4a_5} + f_{a_2a_3c}\, d^{ca_1a_4a_5}\,. \tag{3.16}$$

Using this we get from the formula (2.17) for the reggeizing part

$$\begin{aligned} D_5^{R\,a_1a_2a_3a_4a_5}\big|_{\mathbf{k}_3=0} &= g f_{a_1a_3c} D_4^{R\,ca_2a_4a_5}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_4, \mathbf{k}_5) + g f_{a_2a_3c} D_4^{R\,a_1ca_4a_5}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_4, \mathbf{k}_5) \\ &= g \Big[\quad + \quad \Big] \star D_4^{R\,b_1b_2b_3b_4}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_4, \mathbf{k}_5) \\ &= g \Big[\quad + \quad \Big] \star D_4^{R\,b_1b_2b_3b_4}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_4, \mathbf{k}_5)\,. \end{aligned} \tag{3.17}$$

The Ward identities collected here for the reggeizing parts D_n^R of the n-gluon amplitudes ($n \geq 3$) can be summarized as follows. For vanishing momentum $\mathbf{k}_i$ the momentum part of the amplitude D_n^R reduces to D_{n-1}^R, the momentum arguments being the $(n-1)$ remaining transverse momenta. (Here we again identify $D_3^R = D_3$ since D_3 reggeizes completely, and D_{3-1}^R should be understood as D_2.) In color space the label a_i of the zero-momentum gluon has to be contracted via a $f_{a_j a_i c}$ tensor with the color labels of all gluons j to the left, that is with $j < i$, and these contractions have to be added. The label c has to be taken at the jth position in the amplitude D_{n-1}^R. Since the amplitudes D_{n-1}^R consist of invariant tensors in color space, we can alternatively contract the label a_i with all color labels a_j to the right ($j > i$) with a $f_{a_i a_j c}$ tensor. In the latter case the label c is at the $(j-1)$th position in the amplitude D_{n-1}^R. Casting this into formulae we have

$$D_n^{R\, a_1 \dots a_n}(\mathbf{k}_1, \dots, \mathbf{k}_n)\big|_{\mathbf{k}_i = 0} =$$

$$= g \sum_{j=1}^{i-1} f_{a_j a_i c} D_{n-1}^{R\, a_1 \dots \hat{a}_j \underset{\uparrow j}{c} \dots \hat{a}_i \dots a_n}(\mathbf{k}_1, \dots, \widehat{\mathbf{k}}_i, \dots, \mathbf{k}_n) \tag{3.18}$$

$$= g \sum_{j=i+1}^{n} f_{a_i a_j c} D_{n-1}^{R\, a_1 \dots \hat{a}_i \dots \underset{\uparrow j-1}{c} \hat{a}_j \dots a_n}(\mathbf{k}_1, \dots, \widehat{\mathbf{k}}_i, \dots, \mathbf{k}_n)\,, \tag{3.19}$$

where the hat indicates that the corresponding quantity has to be left out. The formulae include the special cases $\mathbf{k}_1 = 0$ and $\mathbf{k}_n = 0$ as well: the respective sum in (3.18) or (3.19) is empty or it contains $(n-1)$ terms and vanishes due to the condition for invariant tensors (2.11).

In [15] also the reggeizing part D_6^R of the six-gluon has been calculated explicitly. It can also be shown to fulfill the Ward type identities (3.18), (3.19) following the same lines as for up to five gluons.

These Ward type identities for the reggeizing part of the n-gluon amplitude can even be shown to hold for arbitrary n. The reason for this is that they are obtained from the corresponding quark loop $D_{(n;0)}$ by replacing $D_{(2;0)} \to D_2$ as discussed in section 2.2. Due to this construction it is sufficient to prove the identities for $D_{(n;0)}$. The quark loop is the sum of 2^n diagrams. Consider now two of these diagrams that differ in the coupling of the ith gluon. It is coupled to the quark line in one and to the antiquark line in the other diagram. Due to this the two diagrams have opposite sign. But otherwise the momentum structure is the same when setting $\mathbf{k}_i = 0$. The color structure differs by the position of the generator t^{a_i} within the trace of generators around the loop. Starting from one of the two diagrams we can shift t^{a_i} around the loop to the left (or to the right) by iterated use of (2.5). Doing so we come across all gluons j with $j < i$ since the cuts in the amplitude (see figure 1) forbid crossing of t-channel gluons. Therefore this procedure generates exactly the terms needed for (3.18) containing a trace over $(n-1)$ generators contracted with a $f_{a_j a_i c}$. (Although it is a bit tedious, the correct signs can be checked without difficulty.)

The two remaining terms with a trace over n generators cancel due to their different sign mentioned above. The second form of the Ward type identity (3.19) is obtained by shifting the generator t^{a_i} around the loop to the right instead of to the left.

3.2 The amplitudes D_4^I and D_5^I

We now come to examining the Ward type identities for that part of the n-gluon amplitude which is not the superposition of two-gluon amplitudes. This requires $n \geq 4$ since only in this case we have a non-vanishing part D_n^I. On the other hand our knowledge of this part is rather limited: it is only up to $n = 5$ that we know its structure, and even there we do not have an analytic formula for the four-gluon state. Therefore we have to restrict ourselves to D_4^I and D_5^I here.

For D_4^I we already know (see (2.19)) that it vanishes whenever one of the momentum arguments vanishes. The study of the amplitude D_5^I is surprisingly simple. It fulfills Ward type identities very similar to those valid for the reggeizing amplitudes D_n^R, i.e. (3.18) and (3.19). The only difference is well in agreement with what one would naturally expect. Whereas the amplitude D_n^R was reduced to a superposition of D_{n-1}^R's when one momentum was set to zero, D_5^I now reduces to a sum of irreducible D_4^I amplitudes. In detail, we find

$$
\begin{aligned}
&D_5^{I\,a_1\ldots a_5}(\mathbf{k}_1,\ldots,\mathbf{k}_5)\big|_{\mathbf{k}_i=0} = \\
&\quad = g\sum_{j=1}^{i-1} f_{a_j a_i c} D_4^{I\,a_1\ldots\hat{a}_j c\ldots\hat{a}_i\ldots a_5}(\mathbf{k}_1,\ldots,\widehat{\mathbf{k}}_i,\ldots,\mathbf{k}_5) &(3.20)\\
&\quad = g\sum_{j=i+1}^{5} f_{a_i a_j c} D_4^{I\,a_1\ldots\hat{a}_i\ldots c\hat{a}_j\ldots a_n}(\mathbf{k}_1,\ldots,\widehat{\mathbf{k}}_i,\ldots,\mathbf{k}_5) &(3.21)
\end{aligned}
$$

for any $i \in \{1,\ldots,5\}$. Once we have found the amplitude D_5^I in the form (2.20), only two more ingredients are required for the proof. One is the vanishing of D_4^I for vanishing argument (2.19), the other is the defining property (2.11) of invariant $\mathrm{su}(N_c)$ tensors. The latter applies here since the four gluons in the irreducible amplitude D_4^I are in a color singlet state. With these two pieces of information at hand the calculation leading to (3.20) and (3.21) is almost trivial.

3.3 Significance of the Ward type identities

The Ward type identities discussed so far suggest an underlying pattern also valid for higher n-gluon amplitudes. In this section we try to make several conjectures concerning this pattern for higher n. Each of the conjectures applies to a certain part of the amplitudes, for instance to the part of a n-gluon amplitude that reggeizes into two-gluon amplitudes. Since the original integral equations only constrain the full amplitudes D_n we will not be able to prove the conjectures for the different parts separately. The main conjecture we make here is that it is in fact possible to split the amplitudes into different parts in such a way that those parts fulfill the conjectures separately. This main conjecture can of course be tested by investigating amplitudes with more t-channel gluons or by deriving the field theory structure from a different starting point.

Our first conjecture is that the number-changing vertices of the effective field theory should vanish when one of the outgoing gluons has vanishing transverse momentum. This is true [8] for the two-to-four transition vertex $V_{2\to4}$ as well as for the two-to-six gluon transition vertex derived in [15].

Connected with the preceding one is the conjecture that the non-reggeizing parts of the amplitudes, i.e. the parts that cannot be written as superpositions of lower amplitudes, vanish if one of the outgoing gluons has zero momentum. Examples found so far are the two-gluon (BFKL) amplitude D_2 and the irreducible part D_4^I of the four-gluon amplitude.

The final and probably the most important conjecture concerns the reggeizing parts of the amplitudes. If a part of a n-gluon amplitude can be written as a superposition of non-reggeizing parts of lower amplitudes it should satisfy the Ward type identities found in the previous sections.

In order to understand the significance of these conjectures it will be useful to discuss the issue of decomposing the amplitudes into the two parts D_n^R and D_n^I. Initially, the study of the n-gluon amplitudes starts with a set of integral equations for the full amplitudes. The first step is then to *choose* a reggeizing part and to derive a new integral equation for the remaining part. It has turned out that the choice described in section 2.2, namely a superposition of two-gluon amplitudes constructed from the quark loop with n gluons, is singled out because it leads to a particularly simple equation for the remaining part. Only due to this it was possible to discover the field theory structure in the amplitudes. Other choices for the reggeizing part would lead to very complicated equations for the remaining part, and it will in general be difficult to learn something from them. From this point of view our conjectures about the Ward type identities can be understood as conditions for the sensible decomposition of the amplitudes into its different parts which then leads to the identification of the different elements of the effective field theory.

We expect such conditions to become especially helpful already in the course of investigating the six-gluon amplitude. To illustrate the potential significance of the Ward type identities let us briefly discuss the six-gluon amplitude D_6. In the first step, the quark loop offers sufficient inspiration for the sensible choice of a reggeizing part. However, in the six-gluon amplitude a new problem arises. After decomposing the amplitude in the canonical way into reggeizing part and a remaining part, $D_6 = D_6^R + D_6^I$, a new integral equation for D_6^I was found [15]. From its structure it is obvious that a further decomposition is required to fully understand its structure. Namely, the part D_6^I will must contain a part that is the superposition of irreducible four-gluon amplitudes D_4^I, symbolically

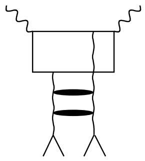
$$D_6^I = D_6^{I,R} + D_6^{I,I} = \sum D_4^I + D_6^{I,I}\,. \tag{3.22}$$

This means that the part $D_6^{I,R}$ is irreducible with respect to the two-gluon amplitude, but it is reducible with respect to the four-gluon amplitude. The part $D_6^{I,I}$ is irreducible with respect to both the two-gluon and the four-gluon amplitude. But in this case we do not have a quark loop suggesting a good choice for $D_6^{I,R}$. Exactly at this point the Ward type identities will be very useful for identifying a correct choice for the reggeizing part in this decomposition. To summarize, we expect roughly the following structure to arise in higher n-gluon amplitudes. There will be irreducible m-gluon compound states for all even m. Based on each of them there will be a hierarchy of reggeizing parts of amplitudes, all of them reggeizing with respect to the same m-gluon compound state. The amplitudes in each of these hierarchies should then obey Ward type identities of the kind discussed above.

4. Reggeization tensors

The preceding section was devoted to the study of the more global interplay of color and momentum structure in the n-gluon amplitudes. Now we turn to more local properties of the color structure. Namely, we will be able to assign to a reggeized gluon a kind of 'quantum number' that specifies its behavior in the process of reggeization. The cleanest environment for studying the mechanism of reggeization is clearly provided by the two-reggeon compound state or BFKL amplitude. To see how higher and higher 'Fock states' of the reggeized gluon occur we will therefore investigate the reggeizing parts D_n^R of the n-gluon amplitudes that consist of superpositions of two-gluon amplitudes. Here we will focus on the color structure of single terms. We will mainly deal with color algebra, and one should be careful in drawing conclusions which go substantially beyond the subject of color structure in this context. At the end we will of course try to interpret the results in a larger context.

The process of reggeization can be viewed in two different ways. To illustrate this let us have a look at the reggeizing part D_4^R of the four-gluon amplitude (2.16). It can be represented diagrammatically as

$$D_4^R(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) = \sum$$  $.\qquad (4.1)$

The sum extends over the different combinations of transverse momenta into two groups as they appear in (2.16). In the picture of t-channel evolution the amplitudes start with the coupling of two reggeized gluons to the photons via a quark loop, then we have the propagation of the two-gluon state in the t-channel, and finally one of the gluons (or both) split — or 'decay' — into two or more gluons. To the splitting of the reggeized gluons belongs a certain color tensor, as given in (2.16) for D_4^R. Viewed from a different angle, we can say that a group of gluons in the reggeizing part D_4^R merges — or 'collapses' — to make up a more composite gluon which then enters the two-gluon compound state from below. We will use both pictures in parallel here and, depending on the context, speak of 'merging' or 'splitting' to mean the very same phenomenon of reggeization.

We will now turn to the case of arbitrary n and consider the reggeizing parts D_n^R of the amplitudes. From these we derive the color structure accompanying the merging of a number of reggeized gluons into a single reggeon. This will lead us in a natural way to a simple classification of the composite reggeons according to their decay properties. Thereby we hope to gain a better understanding of how reggeization works. However, we have to keep in mind that the amplitudes D_n^R constitute only the simplest part of our amplitudes and are derived from the special structure of the quark loop. In a later step we will eventually have to find out whether the reggeization in more complex amplitudes like D_4^I works in the same way, that is whether it is accompanied by the same color tensors as in the two-gluon amplitude. Only then we may speak of a general property of the mechanism of reggeization.

Let us pick one of the terms in the reggeizing part $D_n^R(\mathbf{k}_1, \ldots, \mathbf{k}_n)$. It consists of a two-gluon (BFKL) amplitude D_2 with its two arguments made up from a group of the n momenta each. Let us assume that the first of these groups contains l gluons ($1 \leq l \leq n-1$) and that the other group is made of the remaining m gluons ($m = n - l$). For simplicity, we will further assume that the l gluons in the first group are the first l gluons in the amplitude with momenta $\mathbf{k}_1, \ldots, \mathbf{k}_l$. Other terms in D_n^R with a splitting into l and m gluons are then obtained by permutation of color and momentum labels. We can thus characterize the chosen term by its momentum structure,

$$D_2(\underbrace{\mathbf{k}_1 + \ldots + \mathbf{k}_l}_{l}, \underbrace{\mathbf{k}_{l+1} + \ldots + \mathbf{k}_n}_{m}) \,. \tag{4.2}$$

In this section we will not care about the sign of the special term in D_n^R we consider.[3] We will also neglect the additional factor g^{n-2} that comes with the term above.

To find the color tensor corresponding to (4.2) for arbitrary n we have to remind ourselves of the way the reggeizing parts D_n^R were constructed. The individual terms in D_n^R were obtained by the replacement $D_{(2;0)} \to D_2$ in the quark loop amplitude. The color tensor belonging to (4.2) can therefore be deduced from the corresponding lowest order term in which n gluons are coupled to the quark loop. Specifically, in the

[3] Especially in the case of an odd number of gluons the relative signs of the terms in D_n^R have to be treated with care since the signs change when the order of the color labels in the tensor is reversed.

term of interest in the quark loop there are two contributions: one with the l gluons of the first group coupled to the quark and the other m gluons to the antiquark, the second with quark and antiquark exchanged (as described in section 2.2). The trace in color space taken along the quark loop then gives for the first contribution

$$\mathrm{tr}(t^{b_1}\dots t^{b_l} t^{d_m}\dots t^{d_1})\,. \tag{4.3}$$

Here we have given new color labels to the gluons according to the group they are in. The first l gluons now carry color labels b_i, the m gluons in the second group have now been assigned the color labels d_j such that the connection with the original labels is

$$b_i = a_i \text{ for } i \in \{1,\dots,l\}\,; \qquad d_j = a_{l+j} \text{ for } j \in \{1,\dots,m\}\,. \tag{4.4}$$

The second contribution contains a trace in color space in which the generators occur in reversed order,

$$\mathrm{tr}(t^{d_1}\dots t^{d_l} t^{b_m}\dots t^{b_1})\,. \tag{4.5}$$

The relative sign between the two contributions depends on the total number n of gluons. (This is because the coupling of a gluon to a quark or antiquark in the quark loop effectively differ by a sign.) If n is even, they come with the same sign. So the color tensor we are looking for is

$$d^{b_1\dots b_l d_m\dots d_1} \tag{4.6}$$

as defined in (2.8). If n is odd, the two color traces in (4.3) and (4.5) come with opposite sign and we get a tensor of the form

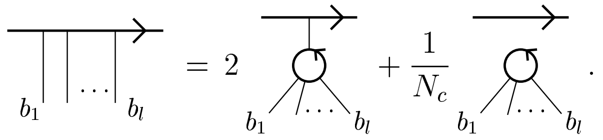
$$f^{b_1\dots b_l d_m\dots d_1} \tag{4.7}$$

as it was defined in (2.9). It should be noted that in both cases the color labels of the one group come in ascending order in the tensor whereas those of the other group have to be taken in reversed order.

Now we make a little digression. It will be useful to have a look at the color structure arising from the successive emission of l gluons off a quark. In color space this process is associated with

$$t^{b_1}\dots t^{b_l} = 2\,\mathrm{tr}(t^{b_1}\dots t^{b_l} t^c) t^c + \frac{1}{N_c}\,\mathrm{tr}(t^{b_1}\dots t^{b_l})\,. \tag{4.8}$$

The proof becomes almost obvious when we write this identity in birdtrack notation,

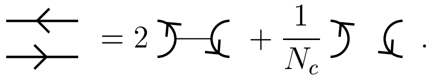


(4.9)

To show this we recall that the decomposition of a quark-antiquark state into a singlet and an adjoint representation (known as the Fierz identity) is

$$\delta^\alpha_\gamma \delta^\delta_\beta = 2(t^a)^\alpha_\beta (t^a)^\delta_\gamma + \frac{1}{N_c}\delta^\alpha_\beta \delta^\delta_\gamma\,, \tag{4.10}$$

where $\alpha, \ldots, \delta$ are color labels in the quark (fundamental) representation. In bird-tracks it becomes



(4.11)

Applying this identity to the right hand side of (4.9) we immediately find the left hand side. Using further the definitions (2.8), (2.9) of d and f tensors we can rewrite this as

$$t^{b_1} \ldots t^{b_l} = \left[d^{b_1 \ldots b_l c} + i f^{b_1 \ldots b_l c}\right] t^c + \frac{1}{2N_c}\left[d^{b_1 \ldots b_l} + i f^{b_1 \ldots b_l}\right]. \tag{4.12}$$

A special case of that general fact is the well-known formula describing the successive emission of two gluons off a quark,

$$t^a t^b = \frac{1}{2}\left[\frac{1}{N_c}\delta_{ab} + (d_{abc} + i f_{abc}) t^c\right]. \tag{4.13}$$

Here we have to have in mind that the conventional normalization of structure constants f_{abc} and d_{abc} differs from our definition of d- and f-tensors with upper indices. Further, we have to use $d^{ab} = \delta_{ab}$ and $f^{ab} = 0$. A seemingly trivial special case of (4.12) is the emission of a single gluon. Namely, if $l = 1$ only the first term on the right hand side gives a contribution, and due to $d^{ab} = \delta_{ab}$ we end up with a trivial identity. Nevertheless, this case is quite important for the consistency of the assignment of 'quantum numbers' we want to carry out.

Coming back to our main problem now, we apply these identities to the color tensor of the term we have picked in D^R_n. The color tensor of that term depends on the total number n of gluons. For even n the color tensor associated with our term is the d-tensor in (4.6). Applying (4.12) now to the first group containing l gluons we arrive at

$$\begin{aligned} d^{b_1 \ldots b_l d_m \ldots d_1} = \; & d^{b_1 \ldots b_l c} d^{c d_1 \ldots d_m} + f^{b_1 \ldots b_l c} f^{c d_1 \ldots d_m} \\ & + \frac{1}{2N_c} d^{b_1 \ldots b_l} d^{d_1 \ldots d_m} + \frac{1}{2N_c} f^{b_1 \ldots b_l} f^{d_1 \ldots d_m}\,. \end{aligned} \tag{4.14}$$

For odd n the color tensor is the f-tensor in (4.7) and here the application of (4.12) to the l gluons in the first group gives

$$\begin{aligned} f^{b_1 \ldots b_l d_m \ldots d_1} = & - d^{b_1 \ldots b_l c} f^{c d_1 \ldots d_m} + f^{b_1 \ldots b_l c} d^{c d_1 \ldots d_m} \\ & - \frac{1}{2N_c} d^{b_1 \ldots b_l} f^{d_1 \ldots d_m} + \frac{1}{2N_c} f^{b_1 \ldots b_l} d^{d_1 \ldots d_m}\,. \end{aligned} \tag{4.15}$$

The relative signs again depend on the order of gluons we started with due to the definition of the f-tensor. We will not pay special attention to this detail here and concentrate on the tensors in the four terms separately.

These two decompositions of the d- and f-tensors contain all possible combinations of even and odd l and m, and the reggeization tensors we are looking for can now be extracted from the two decompositions. Obviously, the tensors describing the splitting in the two groups are correlated. We will now try to assign a kind of quantum number to the two reggeized gluons according to the way the respective reggeons split. This can only make sense if we demand that the two gluons in the two-reggeon state carry the same 'quantum number'. There will be four possible types of reggeons. We first want to fix the type of a reggeized gluon with color label c that does not split. This case is included in the above identities as the splitting into one gluon ($l = 1$ or $m = 1$) via a d-tensor (since we had $d^{ab} = \delta_{ab}$). We want to call this type a reggeized gluon of type f in the adjoint representation. Now we can read off from the first term in (4.14) that such a reggeon decays into an odd number of gluons with a d-tensor. From the first or second term in (4.15) we find that it decays into an even number of gluons with an f-tensor. This assignment is consistent also if both $l > 1$ and $m > 1$.

Secondly, there is also the possibility that a reggeized gluon with color label c splits into an even number of gluons via a d-tensor. We want to call such a reggeized gluon a reggeon of type d in the adjoint representation. From the first lines in (4.14) and (4.15) we find consistently that such a reggeon splits into an odd number of gluons via an f-tensor. An interesting observation is that a reggeon of this type can only occur if it is composite of at least two gluons. Otherwise the corresponding terms in the above identities vanish due to $f^{ab} = 0$. Consequently, such a reggeon can be found in the two-gluon state only if both reggeized gluons decay.

Now we proceed to the last two terms in the right hand sides of (4.14) and (4.15). Here we observe that each of the two groups of reggeons has zero total color charge. The corresponding composite reggeon is obviously in a color singlet state. Although this appears to be somewhat counter-intuitive on first sight we will try to treat these singlet reggeons on equal footing with the reggeons in the adjoint representation discussed before. Again, we can consistently define two different types, d and f. A singlet reggeon of type f splits into an even number of gluons via an f-tensor and into an odd number of gluons via a d-tensor. A d-type singlet reggeon decays into an even number of gluons via a d-tensor and into an odd number of gluons via an f-tensor. Here we do not have a certain decay mode we want to fix as of type d or f like in the case of the adjoint representation. Therefore in the assignment we could as well interchange d and f. Like in the case of the d-type reggeon in the adjoint representation the singlet reggeons are composite of at least two gluons. They cannot occur in a term in D_n^R in which only one of the two reggeized gluons in the two-gluon compound state decays.

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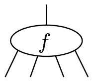
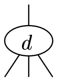
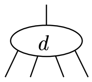
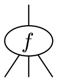
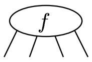
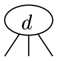
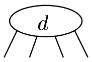
reggeon type	l even	l odd
f, adjoint rep.		
d, adjoint rep.		
f, singlet		
d, singlet		

Table 1: Reggeization tensors as obtained from the reggeizing parts D_n^R

We have been able to extract from the two color identities (4.14) and (4.15) in a consistent way a classification of reggeized gluons in the reggeizing amplitudes D_n^R. The classification is valid for arbitrary n and all possible combinations of numbers l and m that merge into one of the two reggeized gluons entering the two-gluon state. Let us summarize the assignments of reggeon types and the corresponding reggeization tensors in table 1. Here l denotes the number of reggeized gluons that merge into a more composite one. The diagrams for the color tensors are drawn with four and three legs here for illustration only. Of course they represent the tensors of type d or f for an arbitrary number l of gluons as defined in (2.8) and (2.9). For $l \geq 3$ ($l \geq 4$ for the singlet reggeons) the d- and f-tensors in the table can be decomposed further into contractions of symmetric and antisymmetric structure constants. The corresponding formulae for up to $l = 5$ ($l = 6$ for the singlet reggeons) can be found in [15].

In the case of $l = 2$ the d-type reggeons can be interpreted as symmetric in the two gluons, and the f-type reggeons are antisymmetric in the two gluons. This holds in the adjoint representation as well as in the color singlet. However, this interpretation of d (f) as symmetric (antisymmetric) has to be refined for $l \geq 3$ since then the tensors are not completely (anti)symmetric in the l color labels. Instead the symmetry of the d- and f-tensors in individual pairs of gluons becomes more complicated.

In the reggeizing parts D_n^R of the n-gluon amplitudes the different types of reggeons in our classification occur inevitably at the same time, since the tensors (4.14) and (4.15) contain them together. This is not necessarily the case in more complicated parts of the amplitudes that contain a compound state of more than two reggeons. For example, higher amplitudes will contain a part that does not reggeize with respect to the two-gluon amplitude but does reggeize with respect to the four-gluon amplitude. An example is the part D_5^I of the five-gluon amplitude. It is well conceivable that in such amplitudes the four reggeons in the compound state are less correlated than in the two-gluon state and the reggeon types can occur independently.

The example of the part D_5^I of the five-gluon amplitude is, however, not sufficient to clarify the situation. On the one hand, it confirms our classification: the four gluons in D_4^I are of type f in the adjoint representation and one of them in fact splits into two with the tensor that should be expected from our classification. On the other hand, we do not expect the other types of reggeons to appear in D_5^I if our classification is right, since those require two splittings in the whole amplitude. We thus have to go at least to the six-gluon amplitude to study their behavior.

In this respect the f-type reggeon in the adjoint representation plays a special role. It is the only type of reggeon that occurs in D_n^R when only one of the two reggeons decays, i.e. when the other one splits trivially into one gluon. This observation leads us to suspect that the f-type reggeon in the adjoint representation can appear in each possible compound state. For the other types the correlation of the two reggeons in the two-gluon compound state seems to be essential. It seems natural to expect that their behavior in higher compound states is more complicated. Most probably the knowledge of the reggeizing parts D_n^R is not sufficient to fully understand these types.

At present, the classification developed in this section has the status of an observation. We are not able to derive the decay tensors from first principles for arbitrary reggeons in an arbitrary amplitude. Certainly, our findings have to be tested in the investigation of higher n-gluon amplitudes. Especially the universality of the concept outlined here is by no means obvious.

We should note that there is probably a deeper connection with the notion of signature (see for example [18, 4]) to be discovered here. The reggeon of type f in the adjoint representation can be identified as a reggeized gluon with the usual negative signature. The reggeon of type d in the adjoint representation can probably be identified with a reggeized gluon carrying positive signature. For the singlet reggeons, however, the situation is less clear. Presumably it will be necessary to clarify their meaning before the full relation to signature can be established.

5. Conclusions

We have used the generalized leading logarithmic approximation (GLLA) in perturbative QCD to study the reggeization of the gluon at high energies. The reggeized gluon represents an infinite sum of Feynman diagrams in which gluons arrange in such a way that they form a collective excitation of the gauge field with gluon quantum numbers. This mechanism is reflected in the properties of the n-gluon amplitudes arising in the framework of the GLLA. These amplitudes describe the production of

n gluons in the t-channel. They are known explicitly for up to $n = 5$ gluons, and the so-called reggeizing part of the amplitudes can be computed even for arbitrary n. Some parts of the n-gluon amplitudes turn out to be superpositions of lower amplitudes in which a group of gluons merges to form a composite object, i.e. a reggeized gluon. In this way the amplitudes contain information on how the formation of the reggeized gluon takes place.

The reggeization of the gluon is closely related to the emergence of the field theory structure in the n-gluon amplitudes. In this structure there are n-gluon states in the t-channel with even n only which are coupled to each other by transition vertices like the two-to-four or the two-to-six gluon vertex. Accordingly, the amplitudes consist of different parts which are superpositions of two-gluon states, irreducible four-gluon states etc.

We have found Ward type identities relating n-gluon amplitudes with different numbers of gluons. They apply when one of the transverse momenta of the gluons vanishes. These identities place strong constraints on the complicated interplay of the color and momentum structure in the amplitudes which reflects the non-abelian character of QCD. The Ward type identities have been extracted from the known solutions of the n-gluon amplitudes. We have formulated conjectures about their generalization to higher n. An important point is that the Ward type identities apply to different parts of the amplitudes separately. For the reggeizing part (the part that is a superposition of two-gluon amplitudes) we have been able to prove the identities for arbitrary n. We expect them to hold separately also for those parts that are superpositions of irreducible four-gluon and six-gluon amplitudes etc., and we have shown that this is true in the five-gluon amplitude. They will therefore be useful for the difficult problem of separating these parts in higher amplitudes ($n \geq 6$), and consequently in identifying the elements of the effective field theory.

The Ward type identities found here have some similarity to relations for certain families of n-reggeon states which have recently been discovered [19]. Those relations hold for states with a fixed number of reggeized gluons in the large-N_c limit which are described by the BKP equations [5, 20]. The relations in [19] allow one to construct an $n+1$-reggeon state from a state with n reggeons, thus relating states with different numbers of reggeized gluons. Although those relations and our Ward type identities apply to somewhat different amplitudes they might possibly have a common origin.

The reggeizing part of each n-gluon amplitude is a superposition of two-gluon (BFKL) amplitudes in which a number of gluons merges to form a reggeized gluon. We have identified the tensors associated with this process in color space. The color tensors carry the main information about reggeization in the two-gluon amplitude. It was possible to find these tensors for arbitrary numbers of gluons forming a more composite reggeized gluon. It appears possible to assign a sort of quantum number to the reggeized gluon according to how it is formed in color space. The possible types of reggeized gluons include two reggeized gluons in the adjoint representation and two in a color singlet state. We have addressed the possible universality of these color tensors, i.e. the question whether they can also occur in more complicated amplitudes like the irreducible four-gluon state. We expect that one of the reggeon types (the f-reggeon in the adjoint representation) can occur in arbitrary amplitudes. This is confirmed by the explicit solution for the five-gluon amplitude. The behaviour of the other types might be more involved in higher amplitudes because their occurrence requires correlations between different reggeized gluons. It would obviously be very interesting to test the universality of this classification by studying the relevant parts of the six-gluon amplitude, namely the ones involving an irreducible four-gluon state.

With our results we hope to have made a step towards a better understanding not only of reggeization itself but also of the effective field theory structure of unitarity corrections. There is a variety of other approaches to unitarity corrections in high energy QCD (for an extensive list of references see [15]). In many of them the reggeized gluon plays a prominent rôle. It would of course be desirable to confirm the conjectures made in the present paper using one of these approaches. A promising starting point might for example be the effective action approach of [21, 22]. A crucial test of the conjectures will also be the further investigation of the irreducible part of the six-gluon amplitude.

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